

Simple models for granular force networks

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A remarkable feature of static granular matter is the distribution of force along intricate networks. Even regular inter-particle contact networks produce wildly inhomogeneous force networks where certain “chains” of particles carry forces far larger than the mean. In this paper, we briefly review past theoretical approaches to understanding the geometry of force networks. We then investigate the structure of experimentally-obtained granular force networks using a simple algorithm to obtain corresponding graphs. We compare our observations with the results of geometric models, including random bond percolation, which show similar spatial distributions without enforcing vector force balance. Our findings suggest that some aspects of the mean geometry of granular force networks may be captured by these simple descriptions.

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I. INTRODUCTION

Dense granular materials are composed of many particles interacting through multiple, persistent contacts. Without knowing the history of a particular granular assembly, the mobilization of the friction of each contact is indeterminate [1]. Even without friction, the simplest granular materials, such as packings of macroscopically uniform spheres, can still display highly non-uniform networks of contacts — any difference in the size of particles, including microscopic aberrations on the surface of “smooth” spheres, is sufficient to create a complicated contact network [2].

Experiments show that for any given particle within a granular material, the fluctuations in the magnitude of the force carried by that particle can be extremely large [3, 4]. This is the result of long-range force correlations that can be characterized as a *force network* in which some bonds of the contact network carry forces much larger than the mean force on all contacts. As in Fig. 1.2, forces propagate along roughly linear “chains” of contacts. Because the force network depends upon the contact network, the propagation direction and distribution of the chains depends strongly upon the history of the granular material [5, 6].

The overall distribution of forces in a force network is roughly exponential, indicating the possibility for extremely large forces. Since tangential forces are supported only by friction, their distribution falls monotonically, while forces along the chains are peaked at a mean value before falling exponentially [7]. As grains are added, if the force network cannot support the granular material, the material will rearrange until the new contact and force network supports the granular material. For this reason it has been suggested that granular

materials exhibit self-organized criticality, in which the force network behaves as an order parameter in response to displacements of the material [2].

For any configuration of grains (e.g., Fig. 1.1), if the position and shape of each grain is known then a contact network can be generated describing the contacts between the grains. The equations describing a granular material can be grouped into the known equations describing the behavior of the individual grains and the unknown equations describing the contacts between the grains as a function of material assembly and history. These unknown equations can be cast as undetermined relationship between inter-grain forces and the geometry of the contact network [8].

Attempts to resolve the indeterminacy of the constitutive relations have focused upon statistical approaches similar to thermodynamics. Ideally, it might be possible to determine aggregate properties of granular materials that do not depend upon knowing the specifics of the grains to arbitrary precision. Unfortunately there are two problems that have made such an approach non-trivial. First, a granular material at rest is only in a quasi-equilibrium. An arbitrarily small addition of energy can cause irreversible (plastic) rearrangement of the material. This is in part due to the second problem, the lack of an analog to temperature to “thermalize” the system. The actual energy due to temperature is far too small to move grains and, in the static case, there is no other quantity allowing the material to explore phase-space for lower energy states.

Despite these complications, there is a rich tradition of models to understand both the distribution of magnitude and the spatial correlation of forces within granular matter [9, 10, 11]. The applicability of these models to actual granular materials has been limited in part by being restricted to regular lattices of grains. Recent numerical work [12, 13, 14], however, suggests that the spatial distribution of force within irregular arrangements of grains might indeed be characterized through comparison with models with regular geometries.

Typically the force network is determined using a

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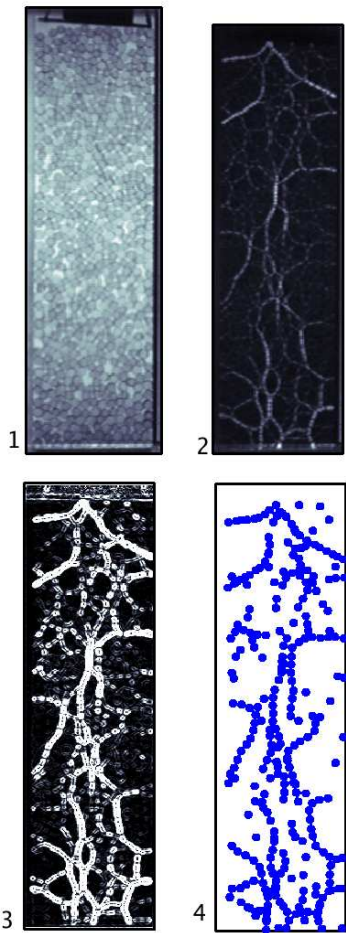


FIG. 1: Analysis of the two-dimensional granular assembly of bi-disperse polymer disks depicted in **1**. The contact network for the disks can be thought as a series of vertices centered at each disk connected by edges wherever two disks are in contact. **2** Because the disks are photoelastic, when they are illuminated between crossed polarizers bright fringes appear in disks under stress allowing force chains to be clearly seen. A load has been applied to the top of the pile causing the creation of additional contacts and force chains supporting the load. **3** We examine the gradient of the stress image as a rough measure of the number of fringes in each disk, which is proportional to the applied force. **4** Finally, we apply an algorithm to the gradient image to find a graph describing the force network.

threshold — for instance, only bonds carrying forces larger than some value, often the mean force, are considered to be part of the force network. In numerical simulations, critical-like transitions from a sparsely distributed network of force chains to a percolating network have been observed as the threshold criteria for including bonds is varied. In one case, the magnitude of the inter-grain mobilization of friction [15] was used and in another that magnitude of inter-grain forces [12]. It is important to note that these thresholded force networks will not be rigid, since they lack bonds that, while car-

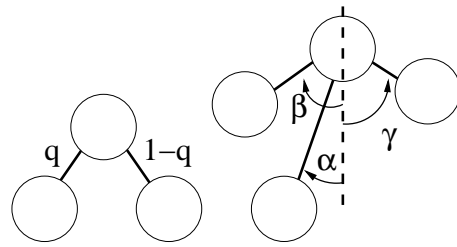


FIG. 2: **Left** In the two-dimensional q -Model, a fraction q of the total weight on a particle on a regular lattice is transferred to one lower neighbor, and the remaining $1 - q$ is transferred to the other. As originally formulated by Coppersmith, et al., the values of q can be drawn randomly from different distributions. In the “critical” case, the value of q is limited to either 0 or 1. **Right** To make rough analogy to the q -Model for more complicated geometries, we use a geometry-based q -Model in which the fraction of a vertex’s weight that is distributed along a downward pointing edge is proportional to the sine of the angle that edge forms with the vertex.

rying only small amounts of force, provide rigidity to the network.

There have been many attempts to model the behavior of force networks, most famously the “ q -Model” of Coppersmith (1996). In the two-dimensional q -Model, depicted in Fig. 2, granular materials are represented as a lattice of sites with two upward neighbors and two downward neighbors. Each site has an intrinsic weight that is added to any weight it receives from the neighbors above it. The total weight on a site is distributed between the two downward neighbors with one receiving a fraction q of the total weight and the other receiving $1 - q$. The q -Model allows the determination of the distribution of weight by working downwards from the top of the lattice. This process can be shown to be diffusive-like in the continuum limit [9].

Diffusive systems, however, do not allow for long-range propagating forces like those seen in force networks. Experiments on small systems have shown that even averaging over multiple force chain networks does not support a diffusive description of granular materials [16]. However, in the case where q is allowed to only take the values of 1 or 0, the q -Model does show chain-like clusters of bonds. This case will be referred to as the “critical q -Model” [9].

In all cases the q -model is a *scalar* model in that the distribution of weight, not force, is being described. The forces exerted upon a particular lattice site do not need to balance, and this in part explains the lack of chain-like structures in all but the critical case. *Vector* models that attempt to include force balance also exist. One of the simplest lattice models that includes force balance is the “tripod” model that, in two-dimensions, assigns each site three downward neighbors, one directly below and two offset to either side by a fixed angle. In the continuum limit this model can give wave-like equations for propagating stress [10]. Additionally, other models have included torque-balance on each site, giving meaning to

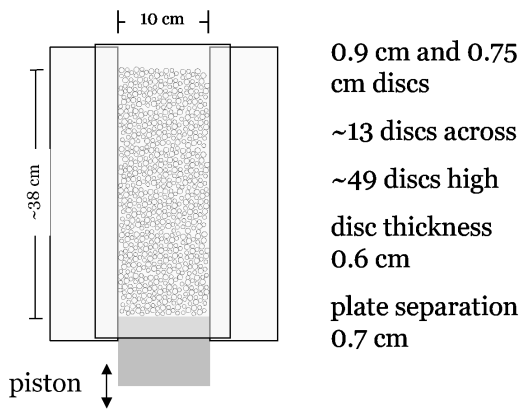


FIG. 3: In our experimental silo for observing granular force networks bi-disperse disks made of a photoelastic polymer are confined vertically to two-dimensions between clear plastic sheets and aluminum spacers serving as sidewalls. The grains rest upon a piston that is lowered slightly to make uniform the mobilization of friction at the walls. The entire silo is illuminated between crossed polarizers, which allows fringes induced by applied stress on each grain to be imaged with a CCD camera.

the finite extent of the grains [11]. Including both force and torque balance allows derivation in simple cases for equations describing force equilibrium as functions of the contact distribution [17].

II. GEOMETRIC APPROACHES TO GRANULAR MATTER

Research into the general behavior of force networks in granular matter has shown that certain properties, including the exponential distribution of large forces, should be expected to arise regardless of the grain-scale model. In this case, some properties of granular materials may be phenomena of the statistics of force networks, and not the micro-mechanics of the grains [18, 19].

This insight has led to approaches to granular materials from the mesoscopic level. By assuming the presence of force chains and describing their behavior, it may be possible to avoid some of the complexity of granular materials and still achieve meaningful results [20, 21]. In particular, network approaches to granular materials may bear fruit because the statistical mechanics of networks in general are beginning to be understood [22].

The critical q -model is similar to random bond percolation on a lattice. Percolation theory provides a framework for understanding how small clusters of connected sites aggregate into larger clusters as a function of the probability that a bond is selected. There are well known values, depending upon the geometry of the lattice, for transitions from loose aggregates of clusters to system-spanning, “infinite” clusters [23]. In the case of the critical q -model, bonds are selected at random from a set

of edges representing the connections between the lattice sites with the restriction that each lattice site has only one load-bearing downward bond. The set of load-bearing bonds can be thought of as a subset of edges that, with the vertices as the lattice sites, forms a sub-graph of the contact network. Since weight at a given vertex is equal to one plus the size of the tree rooted at that vertex, the distribution of cluster sizes gives bounds on the distribution of weights

In disordered granular materials the network of inter-grain contacts may perhaps serve in place of a fixed lattice for percolation. Taking the contact network to be a graph, then the force network may be thought of as the sub-graph consisting of edges with greater than some threshold force.

Properly selecting the edges that comprise the force network requires consideration of vector force balance for each particular network. However, recent work has suggested that granular force networks belong to a specific universality class that can be characterized through analogy to random bond percolation, allowing the distributions of forces in a wide range of isotropic and anisotropic granular systems to be scaled onto a single distribution [12, 13, 14].

In this paper we make experimental observations of the force networks within a granular system and use a simple approach to extract the statistical distributions of graph properties for these force networks. We characterize force networks both in terms of the graph-theoretic properties of degree distribution and cluster-size, as well as q -model inspired properties, including the generalized q -model weight and the size of the cluster resting on a given grain. We then compare our experimentally-observed graphs with those generated by simple geometric models to determine in what ways simple statistical approaches do and do not describe the magnitude and spatial distribution of forces within dense granular matter.

III. METHODOLOGY

We use the experimental setup depicted in Fig. 3 to observe force networks within actual granular matter. We confine 650 bi-disperse (one quarter diameter 0.9 cm and the rest 0.75 cm) photoelastic polymer disks within a vertical, two-dimensional silo. When illuminated between crossed polarizers, photoelastic materials show bright fringes of light in proportion to the stress applied to the material. By taking the gradient of an image of a photoelastic disk along multiple directions we effectively “count” the number of fringes and, using an empirical, linear calibration, can obtain the force distribution throughout the granular material (see Fig. 1.3).

Using the gradient image of a particle force network we convolve with an image representing a single stressed grain to locate the positions of all stressed grains in the image. We then assign an edge between all vertices sep-

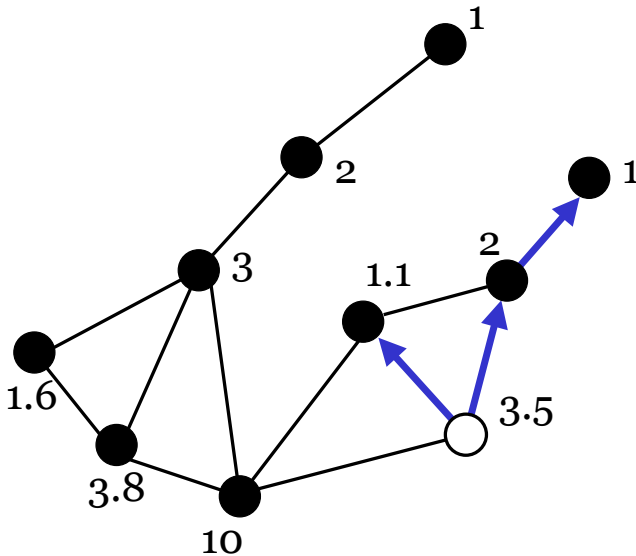


FIG. 4: Illustration of the graph quantities measured. The cluster size $s = 10$ includes all the vertices. The size of the upward tree rooted at the white circle (indicated by directed edges), $t = 4$ includes all vertices connected by upward edges only. The degree of the white node is $k = 3$. The numbers indicate approximate geometric q -Model weights w , determined for each vertex by assigning each an inherent unit weight and distributing the weight among all downward edges in proportion to the sine of the angle each edge makes with the vertex.

arated by a distance in the range of $l/4$ to l where l is a manually-adjusted parameter. The lower range is necessary because some fringe patterns within a single disk produce multiple points when convolved, which has the effect of artificially increases clustering of the graph.

Adding an edge to any grain that already has the threshold value k_{max} edges causes the longest edge to that vertex to be removed. We use the smallest k_{max} that does not significantly alter $P(k)$.

The specific values of the network-finding parameters used depends upon many properties of the images of force networks, including the image resolution and dispersion of the disks. These parameters, in addition to the correlation threshold for locating vertices, must be tuned to produce force network graphs that accurately describe force networks. We assigned edges within the range $l = 13$ pixels and we allowed a maximum of $k_{max} = 5$ edges. As discussed below, the distributions reported here do not change unexpectedly with variation of correlation threshold. The set of vertices and edges found this way are the experimentally obtained force network (see Fig. 1.4).

Using ensembles of force networks we can determine the probability distribution function for the average number of contacts per vertex and number distribution of cluster sizes. We can also record the observed force at each vertex and create an ensemble probability distribution for force.

Figure 4 illustrates several graph quantities that we measure. In addition to cluster size and vertex degree,

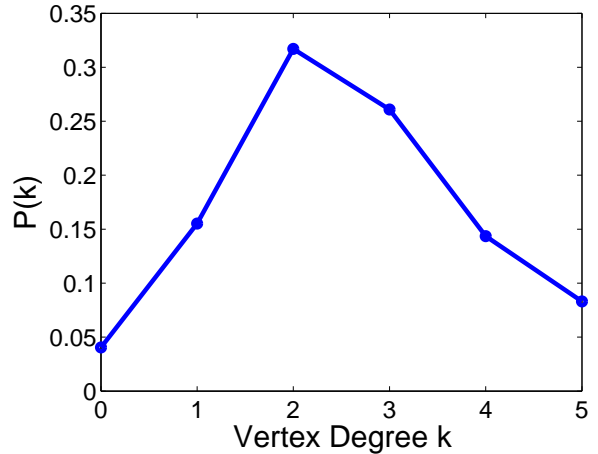


FIG. 5: The distribution of vertex degree peaks at $k = 2$. The lack of a sharp uptick at $k = 5$ indicates that truncating at 5 has not significantly affected the distribution. These results are from analyzing one hundred experimentally-obtained force networks under 56g load.

we measure the size of the upward tree rooted at each vertex. The upward tree is a graph made by treating each edge as a directed edge oriented against gravity. The size of the upward tree corresponds to the number of vertices that are at least partially supported by a given vertex. As illustrated in Fig. 2, weights can be assigned to each vertex and propagated along each downward pointing edge in proportion to the cosine of the angle each edge makes with the vertex. This allows for a q -Model-like weight distribution to be generated for non-lattice geometries. For all of these quantities an aggregate probability density function is calculated.

IV. RESULTS

A. Experimental Results

We analyzed one hundred force networks obtained from experimental images of grains in a vertical silo subject to a 56 g overload that created an approximately uniform distribution of force chains, as in Fig. 1.2. Figure 5 shows the distribution of vertex degrees and, as expected for chain-like structures, vertices most often have two edges. From geometric considerations, the maximum number of contacts for a single disk in a bi-disperse mixture of disks with radii r_{small} and r_{large} is:

$$k_{max} = \pi/2 / \arcsin[r_{small}/2 / (r_{small} + r_{large})] \quad (1)$$

In this case $k_{max} = 6.85$ and there is no probability of seeing a vertex with degree higher than six. Thus, as Fig. 5 shows, the imposed cutoff of $k_{max} = 5$ used in the edge finding routine does not significantly affect the distribution of $P(k)$ since fewer than a tenth of all vertices have degree larger than 4.

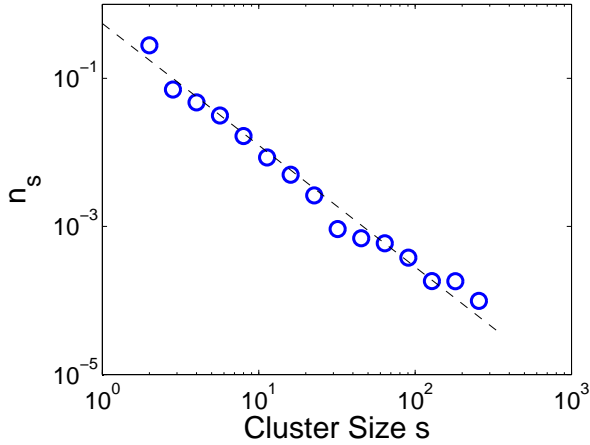


FIG. 6: The distribution of n_s , the number of clusters of size s divided by the total number of clusters, appears to be roughly linear over many decades of probability and two decades of cluster-size (corresponding to the system size) when binned logarithmically. A power-law distribution would support the idea of Roux et al. [2] that the force network may perhaps be considered a critical phenomenon.

The distribution of the number of observed clusters of size s (as a fraction of the total number of clusters) appears to follow a power law, as indicated by Fig. 6. This behavior lends support to the suggestion of Roux, et al. [2] that the force network may be an example of a self-organized critical phenomena. Fits were obtained for regressions to the data (neglecting the end points) for a power law ($n_s = a_1 s^{a_2}$), the product of a power law and an exponential ($n_s = b_1 s^{b_2} e^{b_3 s}$), and a log-normal ($n_s = \frac{c_1}{s} e^{c_2(s-c_3)^2}$) distribution. In all cases, the ratio of the variance predicted by the regression to the observed variance (R^2) is greater than 0.95, indicating likely fits [24]. The fits for the power law-exponential and log-normal distributions, however, both produce positive curvature indicating increasing likelihood with very large cluster size. We believe such distributions are unphysical since increasing probability with increasing cluster size is not normalizable, so that a power law with an exponent $a_2 = -1.6$ seems most likely.

As shown in Fig. 7, the number distribution of upward clusters is shallower ($a_2 = -1.0$) than for whole clusters but still appears to be linear. We expect the distribution of upward trees to generally be shallower than the distribution of clusters since each large cluster contains many upward trees. Regressions to power law, power law-exponential, and log-normal distributions all have $R^2 > 0.85$, but only the fits for the power law-exponential distribution give the observed drop-off with large upward tree size.

Figure 8 shows the probability distribution of intensity of the gradient squared image at the pixel location of each vertex. The intensity corresponds with the force on the disk at that vertex and thus Fig. 8 is an approxi-

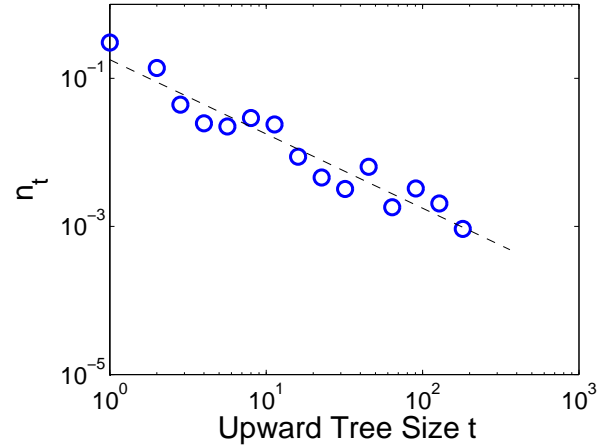


FIG. 7: The logarithmically-binned distribution of n_t , the number of upward clusters of size t divided by the total number of such clusters, appears linear on a log-log plot, similar to the cluster distribution in Fig. 6 but with a shallower slope.

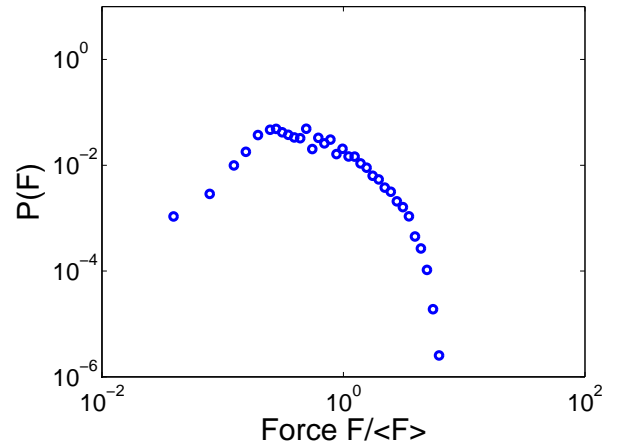


FIG. 8: The force observed at each vertex computed from the gradient is non-monotonic and the greatest probability is for forces below the mean force in the network (indicated by $F/\langle F \rangle = 10^0$).

mate force probability distribution. This non-monotonic distribution peaks below the mean force and then falls off faster than a power law.

We then compare the actual force distribution to what would be predicted using our geometric q -Model, obtaining the distribution in Fig. 9. The q -Model weights are sub-power law but nearly monotonic, unlike the actual force data.

The geometric limit of 6 possible contacts per disk corresponds to 6 potential edges per vertex, however the average number of contacts (coordination number) for a disordered granular material is 4 [2]. Assuming that there are four contacts per vertex, we can calculate a chance p that a bond carrying a large force is present.

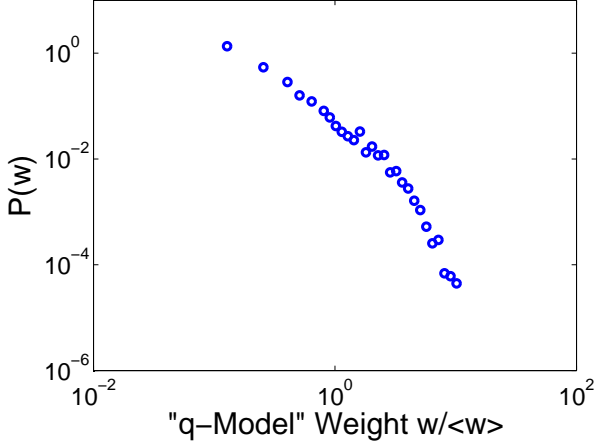


FIG. 9: The probability density function of weight assigned to vertices in the experimentally-obtained force networks by a method similar to the q -Model. Unlike the actual force data, the curve is monotonic except for a slight dip.

For a particular vertex i , $p_i = k_i/4$ and the average value $p \equiv \sum_{i=1}^N p_i/650 = \sum_{i=1}^N k_i/4/650$, where N is the total number of vertices in the force network graph, and 650 is the total number of particles. For the observed networks, $p = 0.2214$.

Additionally, since we expect 4 contacts on average, Fig. 5 indicates that grains in the force network are typically in contact with one or two grains not in the force network.

For each vertex we can also compute the clustering coefficient — how many edges are shared by the neighboring vertices $\binom{4}{2}$ possible edges [25]. Unlike a random network where any vertex may be contact with an unlimited number of other vertices at any position, the choice of our radii and the quasi-two-dimensional geometry of our silo restricts the number of neighbor-neighbor contacts between n neighbors to $n - 1$. We calculate $c = 0.1219$ for the observed force network. This amount of clustering is similar to that seen in scale-free networks that exhibit power law behavior, such as the World-Wide Web and power grids [26], although since we have restricted the number of neighbor-neighbor contacts this is not a perfect analogy.

Finally, we examine the robustness of our results. To obtain force network graphs from our experimental data we convolved our images with a single fringe image kernel that does not necessarily match all arrangements of fringes. We used an acceptability threshold of 0.5, on a scale where 1.0 is a perfect match and 0.0 is the opposite of the kernel. As indicated by Fig. 10, varying the acceptability threshold between 0.4 and 0.6 only slightly affects properties like vertex degree distribution without significantly changing the cluster size power law.

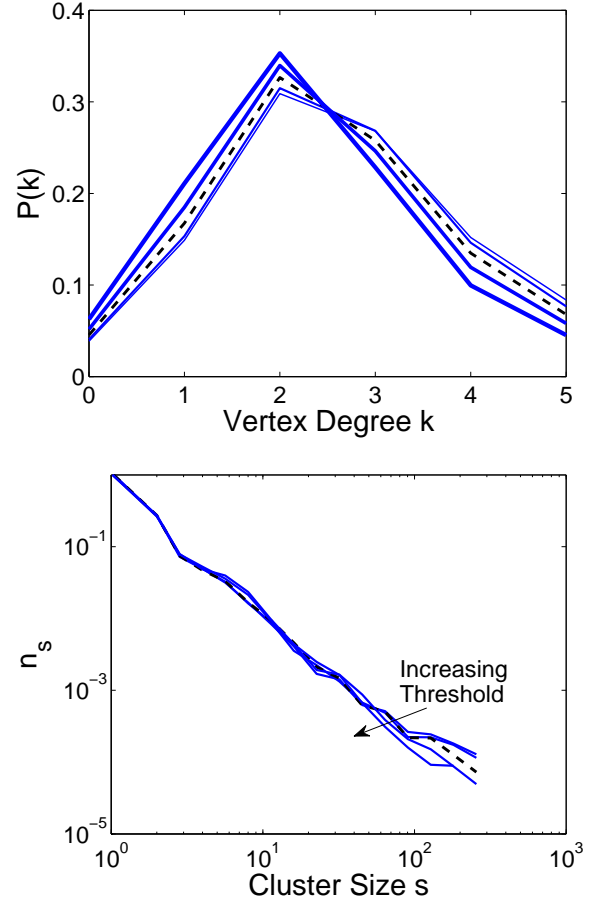


FIG. 10: **Top** Distribution of vertex degree for experimentally obtained force networks as a function of acceptability threshold for convolution with our fringe kernel. The dashed line indicates the distribution for the acceptability threshold of 0.5 used for the analysis in this paper. Lowering the threshold from 0.6 (thickest line) to 0.4 (thinnest line) in increments of 0.05 increases the number of vertices “found” and skews the distribution away from chain-like structures towards higher vertex degree. **Bottom** Despite the change in vertex degree distribution, the distribution of cluster size still indicates a power law with the same slope for all thresholds considered. The dashed line again indicates 0.5.

B. Comparison with Spatial Models

Having characterized actual force networks, it is possible to make comparisons to simple models. Using an ensemble of realizations of the critical q -Model on a 10×40 vertex triangular grid (chosen to reflect the aspect ratio and roughly the number of vertices in our experiments) as in Fig. 11.1, it was possible to determine distribution functions for vertex degree, cluster size, upward tree size and weights assigned by the q -Model.

In addition to the q -Model results, graphs were also constructed via bond percolation. Two regular triangular lattices were considered, one where in addition to downward-angled edges there were horizontal edges

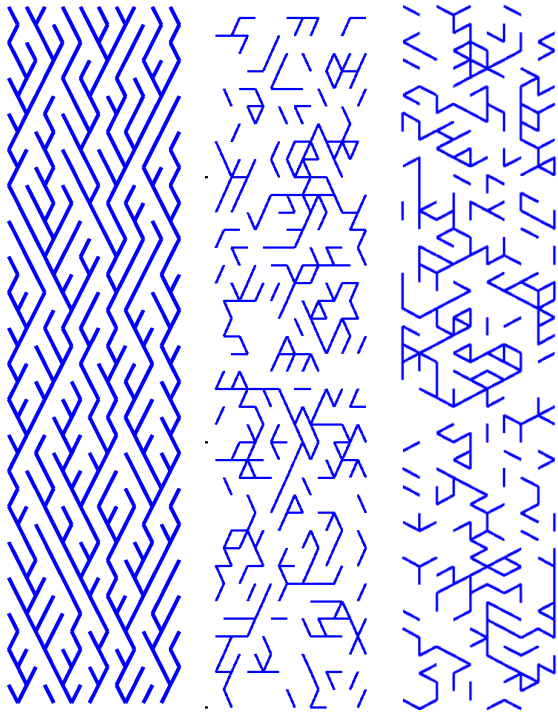


FIG. 11: From left to right: **1** An instance of the q -Model in the critical case. **2** A random network ($p = 0.25$) composed of edges from a triangular lattice oriented horizontally. **3** A random network ($p = 0.25$) composed of edges from a triangular lattice oriented vertically.

(see Fig. 11.2), and one where there were vertical edges (Fig. 11.3). For each realization a subset of these edges were chosen with a probability p to create a randomized graph. With six possible edges per vertex, the triangular lattice is similar to the bi-disperse system though angular distributions are severely restricted.

A range of different p values were examined for both triangular orientations. In addition to the $p = 0.2214$ that was determined experimentally, we examined values of $p = 0.15$ and $p = 0.3$ above and below the estimated p . Finally, we examined $p = 0.5$, a value greater than the percolation threshold, $p_c = 0.3473$, for a system-spanning cluster in an infinite triangular lattice.

We generated 5000 realizations each for the q -model and the two triangular lattice orientations. For the bond percolation graphs both vertex degree and cluster size are independent of whether the basis is oriented horizontally or vertically and so the joint ensemble of 10000 realizations was analyzed.

The distributions of vertex degree for the critical q -Model and triangular lattices with different bond probabilities are shown in Fig. 12. Since the q -Model has to give chain like structures, it is not surprising to see that $P(k)$ peaks at $k = 2$, indicating that most elements are members of a single chain. For a random graph constructed with the bond probability $p = 0.2214$ calculated from our experimental data, the distribution does not

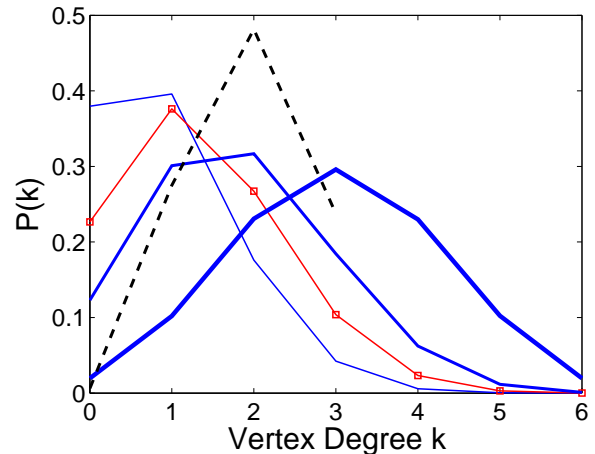


FIG. 12: The probability density function for degree vertex for the critical q -Model and random triangular graphs. The critical q -Model is indicated by a dashed line. The bond percolation results for the experimentally obtained $p = 0.2214$ are indicated by \square . Other bond probabilities, $p = 0.15$, 0.3 and 0.5 , are respectively indicated by lines of increasing thickness. While the q -Model by design gives the expected peak at $k = 2$ — indicating chain-like structures — since in the critical case of the q -Model $k \leq 3$, the best match to Fig. 5 appears to be $p = 3$.

peak appropriately at $k = 2$, but instead at $k = 1$ indicating many pairs of bonds but no clusters. Instead, the peak of the distribution for $p = 0.3$ is most like the experimentally-observed distribution. The distribution for $p = 0.3$ over-represents single bonds when compared with experiments, but the distribution of larger k values matches well.

Also plotted in Fig. 12 is the distribution for $p = 0.15$, below the measured p and not agreeing with the experimentally measured distributions. Finally, the distribution for $p = 0.5$, shows too much skew toward large clusters.

As indicated in Fig. 13, the q -Model gives a distribution (with logarithmic binning) of cluster size that falls linearly ($a_2 = -1.53$) on a log-log plot until broadly peaking at roughly a quarter of the system size (100 vertices). Because we have chosen the critical case of the q -model to ensure long chains of contacts, it is not surprising that we see more large clusters. In contrast, the cluster size distributions for graphs constructed from triangular lattices shown in Fig. 14 are monotonic and roughly linearly, with decreasing slope as p is increased until the critical threshold is crossed, after which the slope for small cluster sizes increases and a sharp peak at the size of the system appears, indicating infinite clusters (similar “bimodal” distributions were observed previously bond percolation simulations of much larger (10^5) systems [27]). The slope for $p = 0.3$ is found to be most similar to that of the experimentally observed networks: $a_2 = -1.77$ with $R^2 \approx 0.97$ compared to $a_2 = -1.6$ for experiment.

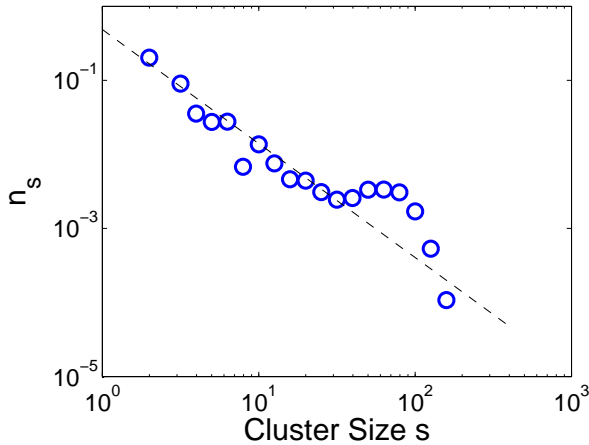


FIG. 13: The distribution of number of clusters of size s for the critical q -Model. On a log-log plot the distribution of smaller clusters is roughly linear (dashed line indicates slope -1.53 found by fitting to linear region), but there is a peak at large cluster sizes since the critical q -Model requires that each site has a downward neighbor.

Figure 15 shows the distribution of “weight” assigned via the q -model for the critical q -Model graphs. The model distribution is more skewed toward large weights than the q -Model-like weight assigned to the experimentally obtained graphs. This is unsurprising since, by construction, the critical q -Model is a connected graph. The distribution of upward trees in Fig. 16 is roughly linear, with nearly the same slope ($a_2 = -1.52$) as cluster size.

In Fig. 17, we have the distribution of weights assigned by the geometric q -Model. None of the models showed the experimentally observed peak for actual force found in Fig. 8. Graphs composed from edges in vertically-aligned lattices — where an additional downward edge is possible — showed slightly broader distributions than those that were composed from horizontally-aligned triangular lattices. In the vertically-aligned lattices, the vertices with weight below the mean are distributed more broadly than in the horizontal case. For both alignments, the distribution for $p = 0.5$ was the most similar to the distribution for graphs obtained from experimental data — this value is much larger than the p measured experimentally. None of these distributions can be said to be a good match for the force or q -Model distributions for experimentally obtained graphs.

As Fig. 18 shows, the distributions of upward trees for graphs below the percolation threshold are similar. For $p = 0.3$ and below, the distributions are well-described by the product of a power law and an exponential ($R^2 > 0.98$) however the coefficients are very different from the experimentally-observed distribution of upward trees. For $p = 0.5$ the distribution is qualitatively similar to the experimental distribution in Fig. 7, but is not well-described by any of the distributions considered (power law, log-normal and power law-exponential). Generally,

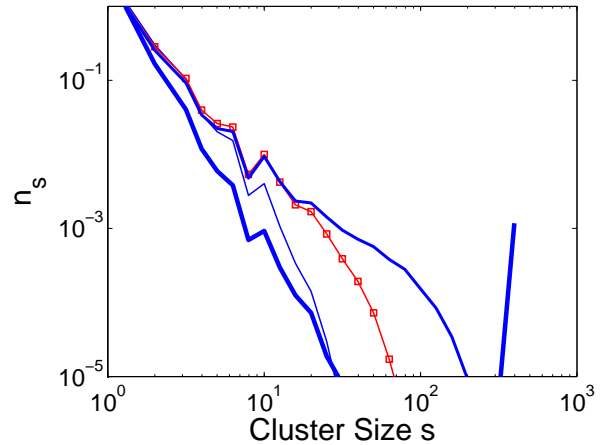


FIG. 14: The distribution of the number of clusters of size s for random triangular graphs. The bond percolation results for the experimentally obtained $p = 0.2214$ are indicated by \square . Other bond probabilities, $p = 0.15, 0.3$ and 0.5 , are respectively indicated by lines of increasing thickness. For all cases there is roughly linear behavior for small to moderate (~ 100) cluster sizes. The transition value to universal clusters is $p_c = 0.3473$ for an infinite triangular lattice and as expected the slope varies non-monotonically, decreasing as p is increased until abruptly dropping at $p = 0.5$ where a second peak at large cluster size indicates that infinite clusters are forming.

when there are vertical edges the distribution of cluster sizes is broader. This is reasonable since the addition of vertical edges makes large trees more likely.

V. CONCLUSION

We have characterized the graphs generated from analyzing experimentally-obtained force networks in granular material. Assuming that there are an average of four contacts per grain, we calculate that the probability that a contact carries a large force is $p = 0.2214$. We observe evidence for a power law scaling of clusters of bonds with large force, possibly indicating a simply-characterized underlying geometry.

When we compare our experimental networks to bond percolation using random sets of edges from triangular lattices, we find that geometric properties of force networks can indeed be matched depending upon the probability of the presence of a bond p . We find that we cannot match all properties of the experimental networks using any one bond probability, including $p = 0.2214$. However, the value of $p = 0.3$ worked well for both the vertex degree and cluster size distribution.

A slightly larger value of p may be an artifact of under-sampling the number of minimally stressed particles that are hard to detect visually but do contribute stressed bonds. It is possible that further refinement of the graph finding algorithm is needed to get the correct value of p

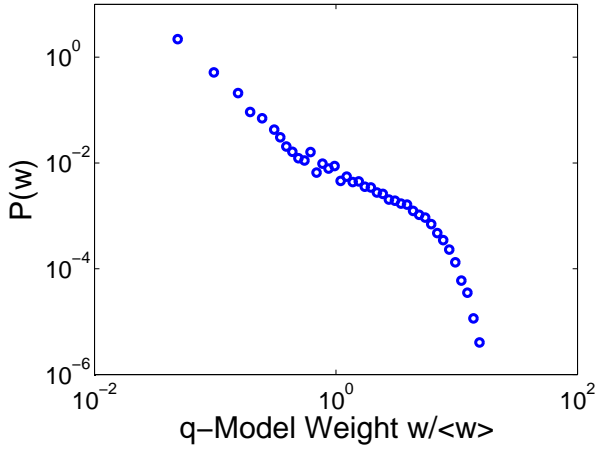


FIG. 15: Probability distribution of weight assigned by the q -Model for the critical q -Model graphs. Because there are far more vertices with weights below the mean, the distribution of q -Model weights is different from both the actual force and geometric q -Model weight distributions for experimentally obtained force networks.

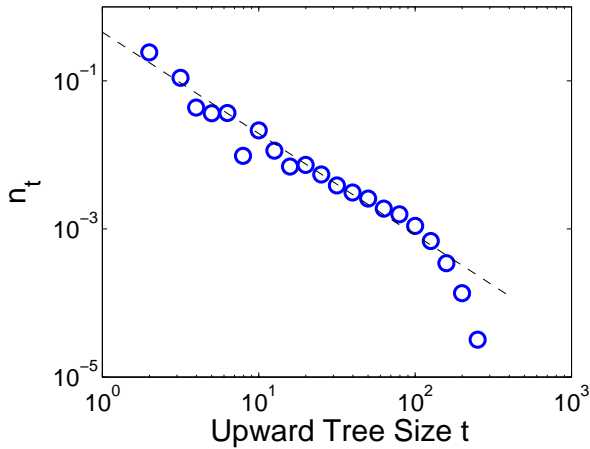


FIG. 16: Number distribution of upward tree size for the critical q -Model.

for actual force networks.

Because of obvious similarities, we have made comparisons with the q -Model and attempted to assign weight to, and determine “force” distribution of, vertices in both q -Model and bond percolation graphs. None of the methods studied provide good agreement with the experimentally-obtained force distribution. This indicates that though graph theoretic approaches may in fact capture the geometry of force networks, the actual force

distributions require more elaborate, vector analysis.

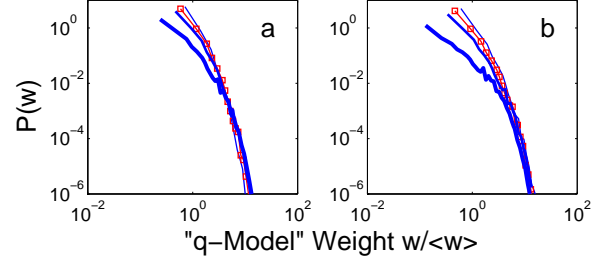


FIG. 17: Probability distribution of geometric q -Model weight for graphs assembled from horizontally- (a) and vertically- (b) aligned triangular lattices. The results for the experimentally obtained $p = 0.2214$ are indicated by \square . Other bond probabilities, $p = 0.15, 0.3$ and 0.5 , are respectively indicated by increasing line thickness. The two alignments differ in the distribution of lower weights. In both cases, only the largest, $p = 0.5$, approaches the distribution observed for experimentally-obtained networks (Fig. 9).

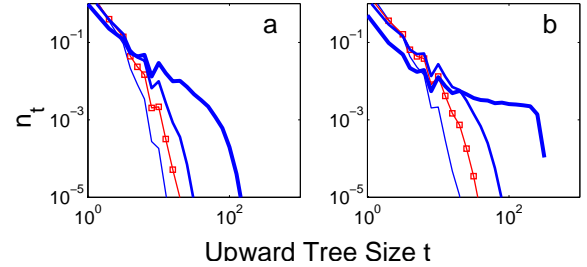


FIG. 18: Distribution of number of upward trees of size t for bond percolation graphs assembled from horizontally- (a) and vertically- (b) aligned triangular lattices. The results for the experimentally obtained $p = 0.2214$ are indicated by \square . Other bond probabilities, $p = 0.15, 0.3$ and 0.5 , are respectively indicated by increasing line thickness. Only when infinite clusters are present ($p = 0.5$) do the distributions change significantly, in both cases giving a distribution that is linear over the same ranges as observed experimentally (Fig. 7).

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